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Homework 1

Question 1. Starting with independent mass and flavor basis/eigenstates of only two dimensions, derive the lepton mixing matrix and demonstrate the length and energy dependence of the oscillatory effect. Show the survivability of ν_μ as a function of length and energy.

We start with describing the independent mass and flavor basis/eigenstates where $\alpha = e, \mu$ and $i = 1, 2$:

$$\begin{aligned} |\nu_\alpha\rangle &= \sum_i U_{\alpha i} |\nu_i\rangle, \\ |\nu_i\rangle &= \sum_\alpha U^*_{\alpha i} |\nu_\alpha\rangle. \end{aligned}$$

This gives complete definitions of the basis/eigenstates for neutrinos in terms of the mass and flavor such that:

$$\langle\nu_\alpha|\nu_\beta\rangle = \langle\nu_i|\nu_j\rangle = \delta_{\alpha\beta} = \delta_{ij} = 1(\text{if } \alpha = \beta \text{ or } i = j),$$

and

$$U = \begin{bmatrix} U_{e1} & U_{e2} \\ U_{\mu 1} & U_{\mu 2} \end{bmatrix}.$$

Again, neutrino flavor eigenstates are a superposition of mass eigenstates which require the propagation of multiple mass/energy eigenstates and is fairly complicated to manipulate because the flavor and mass eigenstates are not closely aligned (like in QCD concerning the CKM matrix). Instead individual neutrino mass basis/eigenstates can easily be manipulated as a plane wave propagating through space-time (where here traditional evolution in time for a standard plane wave also involves spatial coordinates):

$$|\nu_j(t)\rangle = e^{-i(E_j t - \vec{p}_j \cdot \vec{x}))} |\nu_j(0)\rangle.$$

In the ultra-relativistic limit (e.g. where the Lorentz factor, $\gamma >> 1 \implies E \approx p, t \approx L$) and using natural units ($\hbar, c = 1$), the relativistic energy-momentum relation can be approximated (using a Taylor expansion) as:

$$E^2 = \left|p^2\right| + m^2 \implies E_j = \sqrt{|p_j|^2 + m_j^2} \implies p \approx p_j + \frac{p_j^2}{2m} \implies E \approx E_j + \frac{E_j^2}{2m},$$

where the E with no subscript refers to the wave-packet of neutrino flavor. This is allowed because again, neutrino masses are incredibly small. When reinserted back into the plane-wave form for neutrino propagation:

$$|\nu_j(t)\rangle = e^{-i(EL - EL + \frac{m_j^2 L}{2E})} |\nu_j(0)\rangle.$$

$$\implies |\nu_j(t)\rangle = e^{-i\frac{m_j^2 L}{2E}} |\nu_j(0)\rangle.$$

The mass differences (or mass splittings) matter here because even though the masses are small as compared to the energy of neutrinos, the masses relative to each other may not be small and are especially important when acting as a phase in argument of the exponential. *From inspection, it is obvious that if any of the mass splittings are the same, the initial and time evolved state are identical, causing oscillations to vanish! Additionally, the L/E dependence is readily available in the argument of the exponential.*

Now to derive the survivability of ν_μ. To define this, we first need to get to the full evolution of the wave function:

$$\langle\nu_\beta|\nu_\alpha(L)\rangle = \sum_i U^*_{\beta i} \langle\nu_i| \sum_j U_{\alpha j} e^{-i\frac{m_j^2 L}{2E}} |\nu_j\rangle$$

$$= \sum_i \sum_j U^*_{\beta i} U_{\alpha j} e^{-i\frac{m_j^2 L}{2E}} \langle\nu_i|\nu_j\rangle$$

which implies that the general probability of survivability for two flavors can be derived as the wave function squared with its conjugate:

$$P_{(\nu_\alpha \to \nu_\beta)} = |\langle \nu_\beta | \nu_\alpha(L) \rangle|^2 = \left| \sum_j U^*_{\alpha j} U_{\beta j} e^{-i \frac{m_j^2 L}{2E}} \right|^2$$

$$= \sum_j U^*_{\alpha j} U_{\beta j} e^{i \frac{m_j^2 L}{2E}} \sum_k U_{\alpha k} U^*_{\beta k} e^{-i \frac{m_k^2 L}{2E}}$$

and the sums can be combined and limited due to symmetry arguments:

$$= \sum_{j<k} U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k} e^{i \frac{(m_j^2 - m_k^2) L}{2E}}$$

where for survivability $\alpha = \beta = \nu_\mu$. Given the two-dimensional nature of this approximation, there are only two mass states and two flavor states, indicating that the survivability can be derived directly from the oscillation from ν_μ to ν_e. The general description of a two-dimensional rotation matrix is:

$$U = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

which indicates:

$$P_{(\nu_\mu \to \nu_e)} = 2 \sin\theta \cos\theta e^{i \frac{(m_2^2 - m_1^2) L}{2E}}$$

and suggests:

$$P_{(\nu_\mu \to \nu_\mu)} = 1 - \sin^2 2\theta \sin^2 \frac{(m_2^2 - m_1^2) L}{4E}.$$

The values for θ and $m_2^2 - m_1^2$ can be approximated as the best fit value for θ_{12} and the solar mass splitting. ■

Question 2. Take what you learned above and expand to a three-dimensional lepton mixing matrix. Please define the Jarlskog invariant for leptons and show that a degeneracy exists for the mass splittings in consideration of the asymmetry between leptons and anti-leptons. Also show that if any of the mass splittings vanish, the asymmetry vanishes. Finally, derive the probability for the survivability of muon-neutrinos as a function of length and energy.

Using the definition of the lepton mixing matrix from the first problem, the 3 dimensional lepton mixing matrix has entries for three flavor and mass basis/eigenstates and can be written:

$$U = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix}.$$

Returning to the definition of oscillations:

$$P_{(\nu_\alpha \to \nu_\beta)} = |\langle \nu_\beta | \nu_\alpha(L) \rangle|^2 = \left| \sum_j U^*_{\alpha j} U_{\beta j} e^{-i \frac{m_j^2 L}{2E}} \right|^2$$

$$= \sum_{j>k} U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k} e^{i \frac{(m_j^2 - m_k^2) L}{2E}}$$

However this time, the probability is defined in the most general terms of the real and imaginary parts of the matrix product as a consequence of the orthogonality of basis states:

$$P_{(\nu_\alpha \to \nu_\beta)} = \delta_{\alpha\beta} - 4 \sum_{j>k} \mathrm{Re}\{U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k}\} \sin^2 \frac{(m_j^2 - m_k^2) L}{2E} + 2 \sum_{j>k} \mathrm{Im}\{U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k}\} \sin \frac{(m_j^2 - m_k^2) L}{2E}$$

The matter-antimatter asymmetry can be determined by subtracting the probability of oscillation for neutrinos from the probability of oscillation from anti-neutrinos.

$$P_{(\nu_\alpha \to \nu_\beta)} - P_{(\bar{\nu}_\alpha \to \bar{\nu}_\beta)} = 4 \sum_{j>k} \text{Im}\{U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k}\} \sin \frac{(m_j^2 - m_k^2)L}{2E}$$

From this equation the Jarlskog invariant can be readoff as:

$$J = \text{Im}\{U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k}\}.$$

In terms of three flavor oscillations, the matter-antimatter asymmetry can be expressed as:

$$A_{CP} = P_{(\nu_\alpha \to \nu_\beta)} - P_{(\bar{\nu}_\alpha - \bar{\nu}_\beta)} = 16 \sin \frac{(m_2^2 - m_1^2)L}{4E} \sin \frac{(m_3^2 - m_2^2)L}{4E} \sin \frac{(m_3^2 - m_1^2)L}{4E} J \sum_\gamma \epsilon_{\alpha\beta\gamma}.$$

In this form, the degeneracy exists in m_3 as if it is made lighter than $m_{1,2}$ it causes a negative in the argument of two sine functions which cancel each other out and preserve the asymmetry. Additionally if any $m_i = m_j$, one of the arguments of a sine function becomes zero, instantly removing the asymmetry.

Valid forms of the oscillation probability can be found in a few different ways.

1. The most complicated is to start with the full definition of the PMNS matrix and exploiting its similarity to the Euler angles for the definition of a rotation among two different bases in 3 dimensions:

$$U = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{bmatrix},$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. The survival probability for ν_μ can then be derived using the equations above. From here, the enterprising student would have to refer to best fit values for each of the Euler angles ($\theta_{13} = \theta_{reactor}$ and $\theta_{12} = \theta_{solar}$) as well as the leading value for δ_{CP}.

A reasonable approximation can be made by considering the two flavor results from the previous question:

$$P_{(\nu_\mu \to \nu_\mu)} \approx 1 - \sin^2 2\theta_{23} \sin^2 \frac{(m_3^2 - m_2^2)L}{4E}.$$

2. Another equally valid way of deriving the survival probability involves assuming the matrix elements under certain assumptions. The Tri-bimaximal assumption is incompatible with reality (the reactor contribution is small but nonzero) and assumes $\theta_{13} = \theta_{reactor} = 0°$, $\theta_{23} = \theta_{atmospheric} = 45°$, $\theta_{23} = \theta_{solar} = 35.26°$ and gives:

$$U = \begin{bmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{bmatrix}.$$

yielding a similar solution with a slightly different normalization value:

$$P_{(\nu_\mu \to \nu_\mu)} \approx 1 - A \sin^2 \frac{(m_3^2 - m_2^2)L}{4E}.$$

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Question 3. Try to expand this analysis to a four-dimensional lepton mixing matrix. Attempt to the define the appearance probability for neutrinos of the fourth type from muon neutrinos. These are called sterile neutrinos. Derive the probability for the survivability of muon-neutrinos as a function of length, energy, and the magnitude of the fourth mas splitting.

Using the work from the previous questions the PMNS matrix can be expanded with an additional row and column for the presence of a single sterile flavor and additional mass state:

$$U = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} & U_{e4} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} & U_{\mu 4} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} & U_{\tau 4} \\ U_{s1} & U_{s2} & U_{s3} & U_{s4} \end{bmatrix}.$$

In this situation the number of necessary independent angles moves from three to six and three possible sources of matter-antimatter asymmetry exist for Dirac neutrinos. To define this matrix in terms of Euler angles requires the complete definition of each of the rotations and the appropriate order of multiplication however the form of each rotation should be familiar given the definition of the mixing matrix for the two dimensional case:

$$R_{12}(\theta_{12}) = \begin{bmatrix} c_{12} & s_{12} & 0 & 0 \\ -s_{12} & c_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$R_{13}(\theta_{13}) = \begin{bmatrix} c_{13} & 0 & s_{13} & 0 \\ 0 & 1 & 0 & 0 \\ -s_{13} & 0 & c_{13} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$R_{14}(\theta_{14}) = \begin{bmatrix} c_{14} & 0 & 0 & s_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_{14} & 0 & 0 & c_{14} \end{bmatrix},$$

$$R_{23}(\theta_{23}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{23} & s_{23} & 0 \\ 0 & -s_{23} & c_{23} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$R_{24}(\theta_{24}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{24} & 0 & s_{24} \\ 0 & 0 & 1 & 0 \\ 0 & -s_{24} & 0 & c_{24} \end{bmatrix},$$

$$R_{34}(\theta_{34}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_{34} & s_{34} \\ 0 & 0 & -s_{34} & c_{34} \end{bmatrix},$$

with the appropriate multiplication order to define the four dimensional matrix:

$$R = R_{12}(\theta_{12}R_{13}(\theta_{13})R_{14}(\theta_{14})R_{23}(\theta_{23})R_{24}(\theta_{24})R_{34}(\theta_{34})$$

The probability for the appearance of sterile neutrinos can be gleaned from the equation from the previous two questions:

$$P_{(\nu_\alpha \to \nu_\beta)} = \sum_{j<k} U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k} e^{i\frac{(m_j^2 - m_k^2)L}{2E}}.$$

For the ν_s appearance probability from ν_μ the expression can adapted from the 4×4 lepton mixing matrix (with the assumption that the splitting between the first and second mass states is two orders of magnitude smaller compared than the other proposed mass splittings):

$$P_{(\nu_\mu \to \nu_s)} \approx \cos^4\theta_{14}\cos^2\theta_{34}\sin^2 2\theta_{24}\sin^2\frac{(m_4^2 - m_1^2)L}{4E} - \sin^2\theta_{34}\sin^2 2\theta_{23}\sin^2\frac{(m_3^2 - m_1^2)L}{4E}$$
$$+\frac{1}{2}\sin\delta_{24}\sin\theta_{24}\sin 2\theta_{23}\frac{(m_3^2 - m_1^2)L}{4E}.$$

For the survival probability for ν_μ:

$$P_{(\nu_\mu \to \nu_\mu)} \approx 1 - \sin^2 2\theta_{24}\sin^2\frac{(m_4^2 - m_1^2)L}{4E} + 2\sin^2 2\theta_{23}\sin^2 2\theta_{24}\sin^2\frac{(m_3^2 - m_1^2)L}{4E}$$
$$+\frac{1}{2}\sin\delta_{24}\sin\theta_{24}\sin 2\theta_{23}\frac{(m_3^2 - m_1^2)L}{4E}.$$

If the short baseline assumption is utilized ($\frac{(m_3^2-m_1^2)L}{E} << 1$, $\frac{(m_2^2-m_1^2)L}{E} << 1$, with $\frac{(m_4^2-m_1^2)L}{E} >> 1$):

$$P_{(\nu_\mu \to \nu_\mu)} \approx 1 - \sin^2 2\theta_{24} \sin^2 \frac{(m_4^2 - m_1^2)L}{4E}.$$

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Question 4. Plot all ν_μ survivability probabilities as a function of L/E across 7 orders of magnitude starting at 10^{-3} for three different possible values for the mass splittings of the fourth neutrino state versus the first. Those values should be 5, 0.5, and 0.05 eV^2. From this plot make a statement about the best possible location to distinguish which model exists in reality.

(a)

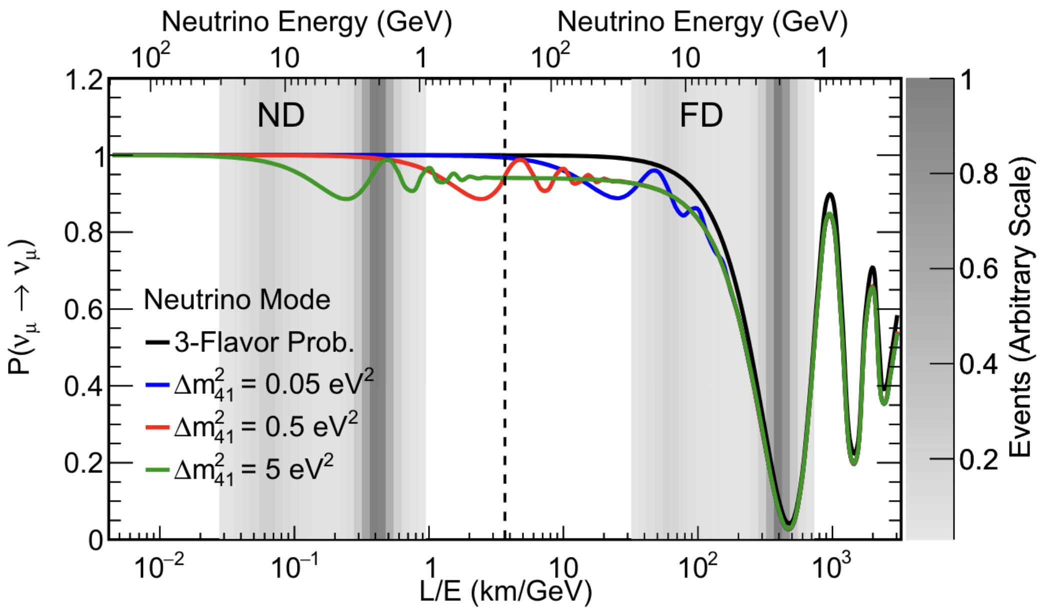

FIGURE 1. ν_μ survivability as a function of L/E and the sterile neutrino mass splitting.

The mass splittings affect the frequency of oscillations and thus the best place to put a detector (or array of detectors) is where their oscillation maxima do not overlap. The largest potential mass proceeds to a steady in disappearance at around 10km/GeV while the midmass value shows clear oscillations across a similar distance that the smallest mass shows a clear decrease. This indicates that a steady state measurement at that distance shows an overall difference in level that would be discernable with the proper precision. Multiple answers to this question exist. ■

REFERENCES

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