Electron Scattering Homework Solution
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1 Homework one: Calculate the elastic scattering kinematics

1.1 General Relativistic equations

For a particle of mass m with kinetic energy K and momentum $\mathbf{P}$, the total energy E can be expressed as

$$E^2 = m^2 + \mathbf{P}^2 \text{ or } E = m + K \tag{1}$$

The magnitude of the momentum $|\mathbf{P}|$ can be calculated from the above two equations

$$m^2 + \mathbf{P}^2 = (m+K)^2 \Rightarrow \mathbf{P}^2 = (m+K)^2 - m^2 \tag{2}$$

Using $a^2 - b^2 = (a-b)(a+b)$

$$\mathbf{P}^2 = |\mathbf{P}|^2 = [(m+K) - m]\,[(m+K) + m] = K(K+2m) \tag{3}$$

Therefore

$$\boxed{|\mathbf{P}| = \sqrt{K(K+2m)}} \tag{4}$$

1.2 Electron side

The incident and outgoing electrons kinematics are (each with mass m_e)

$$\left\{\begin{array}{lcl} E_{inc}^2 &=& m_e^2 + \mathbf{P}_{inc}^2 \\ E_{out}^2 &=& m_e^2 + \mathbf{P}_{out}^2 \end{array}\right. \text{ and } \left\{\begin{array}{lcl} E_{inc} &=& m_e + K_{inc} \\ E_{out} &=& m_e + K_{out} \end{array}\right. \tag{5}$$

From Eqs. 5, one can express the momenta as a function of the kinetic energies:

$$\left\{\begin{array}{lcl} m_e^2 + \mathbf{P}_{inc}^2 &=& (m_e + K_{inc})^2 \\ m_e^2 + \mathbf{P}_{out}^2 &=& (m_e + K_{out})^2 \end{array}\right. \Rightarrow \left\{\begin{array}{lcl} \mathbf{P}_{inc}^2 &=& (m_e + K_{inc})^2 - m_e^2 \\ \mathbf{P}_{out}^2 &=& (m_e + K_{out})^2 - m_e^2 \end{array}\right. \tag{6}$$

$$\left\{\begin{array}{lclcl} \mathbf{P}_{inc}^2 &=& [(m_e + K_{inc}) - m_e][(m_e + K_{inc}) + m_e] &=& K_{inc}(K_{inc} + 2m_e) \\ \mathbf{P}_{out}^2 &=& [(m_e + K_{out}) - m_e][(m_e + K_{out}) + m_e] &=& K_{out}(K_{out} + 2m_e) \end{array}\right. \tag{7}$$

1.3 Elastic Collisions

1.3.1 Nuclei collisions

Let an incident electron of mass m_e with 4-momentum $(E_{inc}, \mathbf{P}_{inc})$ impinging on a nucleus of mass M_A with 4-momentum $(E_A, \mathbf{P}_A)$, leaving with a 4-momentum $(E_{out}, \mathbf{P}_{out})$ and the nucleus with 4-momentum $(E_{A'}, \mathbf{P}_{A'})$. Using the conservation of energy and momentum leads to:

$$e + A \to e' + A' \Rightarrow \begin{cases} E_{inc} + E_A &= E_{out} + E_{A'} \\ \\ \mathbf{P}_{inc} + \mathbf{P}_A &= \mathbf{P}_{out} + \mathbf{P}_{A'} \end{cases} \tag{8}$$

with (from Eq.1)

$$\begin{cases} E_{inc}^2 = m_e^2 + \mathbf{P}_{inc}^2 \quad ; \quad E_A^2 = M_A^2 + \mathbf{P}_A^2 \\ \\ E_{out}^2 = m_e^2 + \mathbf{P}_{out}^2 \quad ; \quad E_{A'}^2 = M_A^2 + \mathbf{P}_{A'}^2 \end{cases} \tag{9}$$

Electrons kinematics From Eq. 8, one can express the energy and momentum of the nucleus after the collision as (noting that in the laboratory frame, the target A is at rest: $|\mathbf{P}_A| = 0$)

$$\begin{cases} E_{A'} &= E_{inc} - E_{out} + E_A \\ \\ \mathbf{P}_{A'} &= \mathbf{P}_{inc} - \mathbf{P}_{out} + \mathbf{P}_A \end{cases} \Rightarrow \begin{cases} E_{A'} &= E_{inc} - E_{out} + M_A \\ \\ \mathbf{P}_{A'} &= \mathbf{P}_{inc} - \mathbf{P}_{out} \end{cases} \tag{10}$$

$$\Rightarrow \begin{cases} E_{A'}^2 &= (E_{inc} - E_{out} + M_A)^2 \\ \\ \mathbf{P}_{A'}^2 &= (\mathbf{P}_{inc} - \mathbf{P}_{out})^2 \end{cases} \tag{11}$$

Expanding the expressions from Eq. 11

$$\begin{cases} E_{A'}^2 &= (E_{inc} - E_{out})^2 + 2\,(E_{inc} - E_{out})\,M_A + M_A^2 \\ \\ \mathbf{P}_{A'}^2 &= \mathbf{P}_{inc}^2 - 2\mathbf{P}_{inc} \cdot \mathbf{P}_{out} + \mathbf{P}_{out}^2 \end{cases} \tag{12}$$

Leading to

$$\begin{cases} \mathbf{P}_{A'}^2 + M_A^2 &= E_{inc}^2 - 2E_{inc}E_{out} + E_{out}^2 + 2\,(E_{inc} - E_{out})\,M_A + M_A^2 \\ \\ \mathbf{P}_{A'}^2 &= \mathbf{P}_{inc}^2 - 2|\mathbf{P}_{inc}||\mathbf{P}_{out}|\cos\Theta + \mathbf{P}_{out}^2 \end{cases} \tag{13}$$

Subtracting the two lines in Eq. 13

$$\cancel{M_A^2} = E_{inc}^2 - 2E_{inc}E_{out} + E_{out}^2 + 2\,(E_{inc} - E_{out})\,M_A + \cancel{M_A^2} - \left(\mathbf{P}_{inc}^2 - 2|\mathbf{P}_{inc}||\mathbf{P}_{out}|\cos\Theta + \mathbf{P}_{out}^2\right) \tag{14}$$

Canceling M_A^2 from both sides and expanding the last term

$$\Rightarrow 0 = E_{inc}^2 - 2E_{inc}E_{out} + E_{out}^2 + 2\,(E_{inc} - E_{out})\,M_A - \mathbf{P}_{inc}^2 + 2|\mathbf{P}_{inc}||\mathbf{P}_{out}|\cos\Theta - \mathbf{P}_{out}^2 \tag{15}$$

Probing nuclear sizes require energies around 100 MeV and beyond. Since the mass of the electron is $m_e = 0.511$ MeV, one can neglect this mass in Eq. 5 at relativistic energies ($\beta = v/c = |\mathbf{P}|/E \gg 0.5$): $E_{inc} = P_{inc} = K_{inc}$ and $E_{out} = P_{out} = K_{out}$

$$\Rightarrow 0 = \cancel{K_{inc}^2} - 2K_{inc}K_{out} + \cancel{K_{out}^2} + 2\,(K_{inc} - K_{out})\,M_A - \cancel{K_{inc}^2} + 2K_{inc}K_{out}\cos\Theta - \cancel{K_{out}^2} \tag{16}$$

Eliminating K_{inc}^2 and K_{out}^2 leads to

$$0 = -2K_{inc}K_{out} + 2\left(K_{inc} - K_{out}\right) M_A + 2K_{inc}K_{out}\cos\Theta \tag{17}$$

which reduces to

$$K_{inc}K_{out}(\cos\Theta - 1) + \left(K_{inc} - K_{out}\right) M_A = 0 \tag{18}$$

Noting that

$$\begin{array}{rcl} \cos^2 x + \sin^2 x & = & 1 \\ \cos^2 x - \sin^2 x & = & \cos 2x \end{array} \Rightarrow 2\sin^2 x = 1 - \cos 2x \Rightarrow -2\sin^2 x = -1 + \cos 2x \tag{19}$$

Replacing x by $\Theta/2$ gives

$$-2\sin^2\Theta/2 = \cos\Theta - 1 \tag{20}$$

Substituting in Eq. 17

$$K_{inc}K_{out}(-2\sin^2\Theta/2) + \left(K_{inc} - K_{out}\right) M_A = 0 \Rightarrow -2K_{inc}K_{out}\sin^2\Theta/2 + \left(K_{inc} - K_{out}\right) M_A = 0 \tag{21}$$

Which can be re-written as

$$\begin{array}{rl} & -2K_{inc}K_{out}\sin^2\Theta/2 + K_{inc}M_A - K_{out}M_A = 0 \\ \Rightarrow & K_{out}\left(-2K_{inc}\sin^2\Theta/2 - M_A\right) + K_{inc}M_A = 0 \end{array} \tag{22}$$

Dividing Eq. 22 by M_A

$$\frac{K_{out}\left(-2K_{inc}\sin^2\Theta/2 - M_A\right)}{M_A} + \frac{K_{inc}M_A}{M_A} = 0 \Rightarrow -K_{out}\left(\frac{2K_{inc}\sin^2\Theta/2}{M_A} + \frac{\cancel{M_A}}{\cancel{M_A}}\right) + K_{inc} = 0 \tag{23}$$

Hence one can solve for K_{out}

$$K_{out}\left(\frac{2K_{inc}\sin^2\Theta/2}{M_A} + 1\right) = K_{inc} \Rightarrow K_{out} = \frac{K_{inc}}{\left(\frac{2K_{inc}\sin^2\Theta/2}{M_A} + 1\right)} \tag{24}$$

Or

$$\boxed{K_{out} = \frac{K_{inc}}{1 + \frac{2K_{inc}}{M_A}\sin^2\Theta/2}} \tag{25}$$

Note that a similar expression can be extracted but solving for K_{inc}

$$\boxed{K_{inc} = \frac{K_{out}}{1 - \frac{2K_{out}}{M_A}\sin^2\Theta/2}} \tag{26}$$

Virtual photon kinematics The virtual photon transferred during the scattering process has energy ω, momentum $\mathbf{q}$ and mass Q^2 which can be expressed as (following Eq. 1)

$$\left\{\begin{array}{lcl} \omega & = & E_{inc} - E_{out} \\ \mathbf{q} & = & \mathbf{P}_{inc} - \mathbf{P}_{out} \\ Q^2 & = & \omega^2 - \mathbf{q}^2 \end{array}\right. \tag{27}$$

Solving for Q^2 and neglecting the electron mass m_e (note that $Q^2 < 0$ since the virtual photon is space-like)

$$\begin{array}{rcl} Q^2 & = & (E_{inc} - E_{out})^2 - (\mathbf{P}_{inc} - \mathbf{P}_{out})^2 \\ & = & K_{inc}^2 - 2K_{inc}K_{out} + K_{out}^2 - K_{inc}^2 + 2K_{inc}K_{out}\cos\Theta - K_{out}^2 \\ & = & -2K_{inc}K_{out} + 2K_{inc}K_{out}\cos\Theta \\ & = & 2K_{inc}K_{out}(\cos\Theta - 1) = 2K_{inc}K_{out}(-2\sin^2\Theta/2) \end{array} \tag{28}$$

Which leads to

$$\boxed{Q^2 = -4K_{inc}K_{out}\sin^2\Theta/2} \tag{29}$$

Solving for ω using Eq. 21

$$-2K_{inc}K_{out}\sin^2\Theta/2 + (K_{inc} - K_{out})\,M_A = 0 \Rightarrow \frac{Q^2}{2} + \omega M_A = 0 \tag{30}$$

Re-arranging to obtain ω

$$\frac{Q^2}{2} = -\omega M_A \Rightarrow \boxed{\omega = \frac{-Q^2}{2M_A}} \tag{31}$$

Solving for $\mathbf{q}$ using Eq. 27

$$|\mathbf{q}|^2 = (\mathbf{P}_{inc} - \mathbf{P}_{out})^2 = K_{inc}^2 - 2K_{inc}K_{out}\cos\Theta + K_{out}^2 \tag{32}$$

Or

$$\boxed{q = \sqrt{K_{inc}^2 - 2K_{inc}K_{out}\cos\Theta + K_{out}^2}} \tag{33}$$

1.3.2 Nucleon collisions or quasi-elastic scattering

Quasi-elastic scattering is an elastic scattering process when the electron penetrates the nucleus and elastically scattered off a nucleon. Since nucleons are packed inside a tight volume (e.g., the nucleus) they are not at rest and move at relativistic speeds: the scattering occurs with nucleons having a momentum distribution centered around some mean value.

Let an incident electron of mass m_e with 4-momentum $(E_{inc}, \mathbf{P}_{inc})$ impinging on a nucleon of mass M_N with 4-momentum $(E_N, \mathbf{P}_N)$, leaving with a 4-momentum $(E_{out}, \mathbf{P}_{out})$ and the nucleon with 4-momentum $(E_{N'}, \mathbf{P}_{A'})$. Using the conservation of energy and momentum leads to:

$$e + N \rightarrow e' + N' \Rightarrow \left\{\begin{array}{lcl} E_{inc} + E_N & = & E_{out} + E_{N'} \\ \\ \mathbf{P}_{inc} + \mathbf{P}_N & = & \mathbf{P}_{out} + \mathbf{P}_{N'} \end{array}\right. \tag{34}$$

with

$$\left\{\begin{array}{lclclcl} E_{inc}^2 & = & m_e^2 + \mathbf{P}_{inc}^2 & ; & E_N^2 & = & M_N^2 + \mathbf{P}_N^2 \\ \\ E_{out}^2 & = & m_e^2 + \mathbf{P}_{out}^2 & ; & E_{N'}^2 & = & M_N^2 + \mathbf{P}_{N'}^2 \end{array}\right. \tag{35}$$

Electrons kinematics Following the same procedure as in section 1.3.1 leads to

$$\begin{cases} E_{N'}^2 &=& (E_{inc} - E_{out} + E_N)^2 \\ \\ \mathbf{P}_{N'}^2 &=& (\mathbf{P}_{inc} - \mathbf{P}_{out} + \mathbf{P}_N)^2 \end{cases} \tag{36}$$

Expanding the expressions from Eq. 36

$$\begin{cases} E_{N'}^2 &=& (E_{inc} - E_{out})^2 + 2\,(E_{inc} - E_{out})\,E_N + E_N^2 \\ \\ \mathbf{P}_{N'}^2 &=& (\mathbf{P}_{inc} - \mathbf{P}_{out})^2 + 2\,(\mathbf{P}_{inc} - \mathbf{P}_{out}) \cdot \mathbf{P}_N + \mathbf{P}_N^2 \end{cases} \tag{37}$$

Therefore

$$\begin{cases} \mathbf{P}_{N'}^2 + M_N^2 &=& E_{inc}^2 - 2E_{inc}E_{out} + E_{out}^2 + 2\,(E_{inc} - E_{out})\,E_N + E_N^2 \\ \\ \mathbf{P}_{N'}^2 &=& \mathbf{P}_{inc}^2 - 2|\mathbf{P}_{inc}||\mathbf{P}_{out}|\cos\Theta + \mathbf{P}_{out}^2 + 2\,(\mathbf{P}_{inc} - \mathbf{P}_{out}) \cdot \mathbf{P}_N + \mathbf{P}_N^2 \end{cases} \tag{38}$$

Subtracting the two lines in Eq. 38 leads to

$$\begin{array}{rcl} M_N^2 &=& E_{inc}^2 - 2E_{inc}E_{out} + E_{out}^2 + 2\,(E_{inc} - E_{out})\,E_N + E_N^2 \\ &-& \mathbf{P}_{inc}^2 + 2|\mathbf{P}_{inc}||\mathbf{P}_{out}|\cos\Theta - \mathbf{P}_{out}^2 - 2\,(\mathbf{P}_{inc} - \mathbf{P}_{out}) \cdot \mathbf{P}_N - \mathbf{P}_N^2 \end{array} \tag{39}$$

Consequently

$$\begin{array}{rcl} M_N^2 &=& (E_{inc}^2 - \mathbf{P}_{inc}^2) - 2E_{inc}E_{out} + (E_{out}^2 - \mathbf{P}_{out}^2) + 2\,(E_{inc} - E_{out})\,E_N + (E_N^2 - \mathbf{P}_N^2) \\ &+& 2|\mathbf{P}_{inc}||\mathbf{P}_{out}|\cos\Theta - 2\,(\mathbf{P}_{inc} - \mathbf{P}_{out}) \cdot \mathbf{P}_N \end{array} \tag{40}$$

$$\begin{array}{rcl} \cancel{M_N^2} &=& m_e^2 - 2E_{inc}E_{out} + m_e^2 + 2\,(E_{inc} - E_{out})\,E_N + \cancel{M_N^2} \\ &+& 2|\mathbf{P}_{inc}||\mathbf{P}_{out}|\cos\Theta - 2\,(\mathbf{P}_{inc} - \mathbf{P}_{out}) \cdot \mathbf{P}_N \end{array} \tag{41}$$

Canceling M_N^2 from both sides and expanding the last term

$$0 = m_e^2 - 2E_{inc}E_{out} + m_e^2 + 2\,(E_{inc} - E_{out})\,E_N + 2|\mathbf{P}_{inc}||\mathbf{P}_{out}|\cos\Theta - 2\,(\mathbf{P}_{inc} - \mathbf{P}_{out}) \cdot \mathbf{P}_N \tag{42}$$

Neglecting the mass of the electron as discussed in section 1.3.1

$$0 = -2K_{inc}K_{out} + 2\,(K_{inc} - K_{out})\,E_N + 2|\mathbf{P}_{inc}||\mathbf{P}_{out}|\cos\Theta - 2\,(\mathbf{P}_{inc} - \mathbf{P}_{out}) \cdot \mathbf{P}_N \tag{43}$$

Substituting $\omega = K_{inc} - K_{out}$ and $\mathbf{q} = \mathbf{P}_{inc} - \mathbf{P}_{out}$

$$\begin{array}{rcl} 0 &=& -2K_{inc}K_{out} + 2\omega E_N + 2K_{inc}K_{out}\cos\Theta - 2\mathbf{q} \cdot \mathbf{P}_N \\ &=& 2K_{inc}K_{out}\,(\cos\Theta - 1) + 2\omega E_N - 2\mathbf{q} \cdot \mathbf{P}_N \\ &=& -4K_{inc}K_{out}\sin^2\Theta/2 + 2\omega E_N - 2\mathbf{q} \cdot \mathbf{P}_N \\ &=& Q^2 + 2\omega E_N - 2\mathbf{q} \cdot \mathbf{P}_N \end{array} \tag{44}$$

Solving for ω

$$Q^2 + 2\omega E_N - 2\mathbf{q} \cdot \mathbf{P}_N = 0 \Rightarrow 2\omega E_N = Q^2 + 2\mathbf{q} \cdot \mathbf{P}_N \tag{45}$$

which gives

$$\boxed{\omega = \frac{Q^2}{2E_N} + \frac{\mathbf{q} \cdot \mathbf{P}_N}{E_N}} \tag{46}$$

The mass of a nucleon M_N is of the order of 1 GeV ($M_{proton} = 938$ MeV and $M_{neutron} = 939$ MeV). Making the approximation that $|\mathbf{P}_N| \ll M_N$, one is left with $E_N \simeq M_N$. Therefore

$$\omega \simeq \frac{Q^2}{2M_N} + \frac{\mathbf{q}\cdot\mathbf{P}_N}{M_N} \tag{47}$$

We can define the mean binding energy per nucleon $\bar{\varepsilon}$ to be

$$\bar{\varepsilon} = \frac{\mathbf{q}\cdot\mathbf{P}_N}{M_N} \Rightarrow \omega \simeq \frac{Q^2}{2M_N} + \bar{\varepsilon} \tag{48}$$

The energy transferred can be re-written as

$$\boxed{\omega \simeq \frac{Q^2}{2M_N^*}} \tag{49}$$

where M_N^* is the effective mass of a nucleon that includes nuclear effect.

Virtual photon kinematics They are similar to the ones define in section 1.3.1.

2 Homework 2: Form factor calculation

Assuming a known density for a homogeneous sphere with Z protons in a volume $V = \frac{4}{3}\pi R^3$

$$\rho_0(r)\begin{cases} Z/(\frac{4}{3}\pi R^3) & \text{for } r \leq R \\ 0 & \text{for } r > R \end{cases} \tag{50}$$

One can normalize the charge to 1

$$\rho(r)\begin{cases} 1/(\frac{4}{3}\pi R^3) & \text{for } r \leq R \\ 0 & \text{for } r > R \end{cases} \tag{51}$$

The form factor $F(\mathbf{q}^2)$ is defined as

$$F(\mathbf{q}^2) = 4\pi \int \rho(r)\frac{\sin(|\mathbf{q}|r/\hbar)}{|\mathbf{q}|r/\hbar} r^2 dr \tag{52}$$

Fron Eq.51

$$\begin{aligned} F(\mathbf{q}^2) &= 4\pi \int_0^R \rho(r)\tfrac{\sin(|\mathbf{q}|r/\hbar)}{|\mathbf{q}|r/\hbar} r^2 dr = 4\pi \int_0^R \tfrac{1}{\frac{4}{3}\pi R^3}\tfrac{\sin(|\mathbf{q}|r/\hbar)}{|\mathbf{q}|r/\hbar} r^2 dr \\ &= \cancel{4\pi}\tfrac{3}{\cancel{4\pi}R^3}\int_0^R \tfrac{\sin(|\mathbf{q}|r/\hbar)}{|\mathbf{q}|r/\hbar} r^2 dr = \tfrac{3}{R^3}\int_0^R \tfrac{\sin(|\mathbf{q}|r/\hbar)}{|\mathbf{q}|r/\hbar} r^2 dr \end{aligned} \tag{53}$$

Let define

$$x = \frac{|\mathbf{q}|r}{\hbar} \rightarrow r = \frac{x\hbar}{|\mathbf{q}|} \rightarrow dr = \frac{\hbar}{|\mathbf{q}|}dx \tag{54}$$

The limits of the integral become

$$\begin{cases} r = 0 & \rightarrow & x = 0 \\ r = R & \rightarrow & x_{max} = \alpha = \frac{|\mathbf{q}|R}{\hbar} \end{cases} \tag{55}$$

Substituting in Eq. 53

$$\begin{aligned} F(\mathbf{q}^2) &= \tfrac{3}{R^3}\int_0^\alpha \tfrac{\sin x}{x}\left(\tfrac{x\hbar}{|\mathbf{q}|}\right)^2 \tfrac{\hbar}{|\mathbf{q}|}dx \\ &= \tfrac{3}{R^3}\int_0^\alpha \tfrac{\sin x}{x}x^2\left(\tfrac{\hbar}{|\mathbf{q}|}\right)^2 \tfrac{\hbar}{|\mathbf{q}|}dx \\ &= \tfrac{3}{R^3}\int_0^\alpha \tfrac{\sin x}{\not x}x^{\not 2}\left(\tfrac{\hbar}{|\mathbf{q}|}\right)^3 dx \\ &= \tfrac{3}{R^3}\left(\tfrac{\hbar}{|\mathbf{q}|}\right)^3 \int_0^\alpha x\sin x\; dx \end{aligned} \tag{56}$$

The integral can be solved independently

$$\int x\sin x\; dx = \sin x - x\cos x \tag{57}$$

Substituting in Eq. 56

$$\begin{aligned} F(\mathbf{q}^2) &= \tfrac{3}{R^3}\left(\tfrac{\hbar}{|\mathbf{q}|}\right)^3 \big[\sin x - x\cos x\big]_0^\alpha \\ &= 3\left(\tfrac{\hbar}{|\mathbf{q}|R}\right)^3 \big[\sin\alpha - \alpha\cos\alpha - (\sin 0 - 0\cos 0)\big] \\ &= 3\left(\tfrac{1}{\alpha}\right)^3 \big[\sin\alpha - \alpha\cos\alpha\big] \end{aligned} \tag{58}$$

Therefore

$$\boxed{F(\mathbf{q}^2) = 4\pi\int \rho(r)\frac{\sin(|\mathbf{q}|r/\hbar}{|\mathbf{q}|r/\hbar}r^2dr = \frac{3}{\alpha^3}\big[\sin\alpha - \alpha\cos\alpha\big]} \tag{59}$$